Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – discrete mathematics

Wednesday 9 May 2018 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[50 marks]**.

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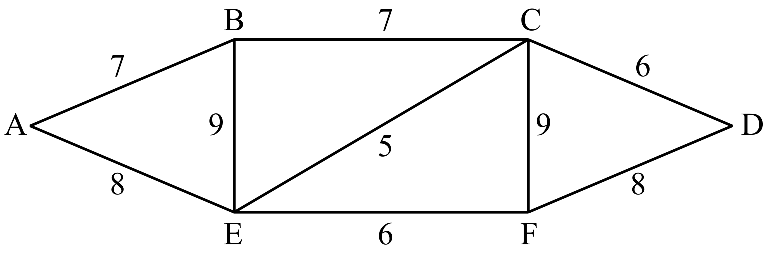
Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 10]

Consider the following weighted graph G.



(a) State what feature of G ensures that

(i) G has an Eulerian trail;

(ii) G does not have an Eulerian circuit. [2]

(b) Write down an Eulerian trail in G. [2]

(c) (i) State the Chinese postman problem.

(ii) Starting and finishing at B, find a solution to the Chinese postman problem for G.

(iii) Calculate the total weight of the solution. [6]

2. [Maximum mark: 8]

(a) State Fermat's little theorem. [2]

(b) Consider the linear congruence $ax \equiv b (\mathrm{mod}\, p)$ where $a, b, p, x \in \mathbb{Z}^+$, p is prime and a is not a multiple of p.

(i) Use Fermat's little theorem to show that $x \equiv a^{p-2} b (\mathrm{mod}\, p)$.

(ii) Hence solve the linear congruence $5x \equiv 7 (\mathrm{mod}\, 13)$. [6]

3. [Maximum mark: 11]

Consider the complete bipartite graph $\kappa_{3,3}$.

(a) (i) Draw $\kappa_{3,3}$.

(ii) Show that $\kappa_{3,3}$ has a Hamiltonian cycle.

(iii) Draw $\kappa_{3,2}$ and explain why it does not have a Hamiltonian cycle. [4]

(b) (i) In the context of graph theory, state the handshaking lemma.

(ii) Hence show that a graph G with degree sequence 2, 3, 3, 4, 4, 5 cannot exist. [3]

Let T be a tree with v vertices where $v \geq 2$.

(c) Use the handshaking lemma to prove that T has at least two vertices of degree one. [4]

4. [Maximum mark: 6]

(a) Show that $\gcd(4k+2, 3k+1) = \gcd(k-1, 2)$, where $k \in \mathbb{Z}^+$, $k > 1$. [4]

(b) State the value of $\gcd(4k+2, 3k+1)$ for

(i) odd positive integers k ;

(ii) even positive integers k. [2]

5. [Maximum mark: 15]

The Fibonacci sequence can be described by the recurrence relation $f_{n+2} = f_{n+1} + f_n$ where $f_0 = 0$, $f_1 = 1$.

(a) Write down the auxiliary equation and use it to find an expression for f_n in terms of n. [7]

It is known that $\alpha^2 = \alpha + 1$ where $\alpha = \dfrac{1+\sqrt{5}}{2}$.

(b) For integers $n \geq 3$, use strong induction on the recurrence relation $f_{n+2} = f_{n+1} + f_n$ to prove that $f_n > \alpha^{n-2}$. [8]